

MOMENTS OF ZETA AND CORRELATIONS OF DIVISOR-SUMS: I

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ABSTRACT. We examine the calculation of the second and fourth moments and shifted moments of the Riemann zeta-function on the critical line using long Dirichlet polynomials and divisor correlations. Previously this approach has proved unsuccessful in computing moments beyond the eighth, even heuristically. A careful analysis of the second and fourth moments illustrates the nature of the problem and enables us to identify the terms that are missed in the standard application of these methods.

1. INTRODUCTION

We revisit the long-standing problem of determining the moments of the Riemann zeta-function on the critical line from a somewhat new perspective. On one hand, we have the detailed conjectures of [KS, CFKRS] which predict very precisely the outcome of any moment calculation. On the other hand, we have the conventional approaches of analytic number theorists via Dirichlet polynomial approximations to $\zeta(s)$ or $\zeta(s)^k$ which proceed by refinements of the approximate Parseval formula for Dirichlet polynomials (i.e. the Montgomery-Vaughan formula). We would like to bring these approaches closer together. In particular, the Dirichlet polynomial approach completely fails when considering the 10th or higher moment of $\zeta(s)$ on the critical line: it predicts negative values when the moments are clearly non-negative. We want to understand and rectify, at least from a heuristic perspective, this failing.

Our aim here is to illustrate the nature and cause of this failing by a careful analysis of the second and fourth moments. We take the standard approach of approximating the zeta function by long Dirichlet polynomials and computing moments using divisor correlations, but depart in using longer polynomials than is normal.

The approximate functional equation expresses the zeta function at a height T on the critical line by Dirichlet polynomials of length of the order of $T^{1/2}$. This allows all principal terms (called, in [CFKRS], the moment polynomials) in the second [I, A] and fourth [HB]

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moments to be determined. This approach extends heuristically to the sixth moment, and to calculating the leading-order in the asymptotics of the eighth moment; see, for example, [CG]. For moments beyond the eighth, it leads to answers that are clearly incorrect, in that they are negative. In order to understand this problem better, we here examine the calculations for the second and fourth moments when we use longer Dirichlet polynomials; specifically, we take the length to be substantially larger than T (and so do not utilize the approximate functional equation). We find that in this case the success of the standard methods when applied to the second moment rests on a particular identity, and that they fail when applied to the fourth moment by missing certain key terms. We believe that this sheds new light on the moment problem.

2. SECOND MOMENT

We begin with the simplest case, that of the second moment of $\zeta(s)$; even in this case our perspective reveals some new information. Our starting point is the theorem of Ingham [I] that

$$\int_0^T \zeta(s + \alpha) \zeta(1 - s + \beta) dt = \int_0^T \left(\zeta(1 + \alpha + \beta) + \left(\frac{t}{2\pi} \right)^{-\alpha - \beta} \zeta(1 - \alpha - \beta) \right) dt + O(T^{1/2+\epsilon})$$

uniformly for small α and β , where $s = 1/2 + it$. Sandro Bettin [B] has proven that, remarkably, the same formula holds for a much larger range of α and β namely $|\Re \alpha|, |\Re \beta| < 1/4$ and $|\Im \alpha|, |\Im \beta| \ll T$.

We compare this result with what one gets when using the work of Goldston-Gonek [GG] on the mean-square of long Dirichlet polynomials. We refer to Corollary 1 of [GG] together with example 1, which is about the case $\alpha = \beta = 0$ of Ingham's theorem. In particular we approximate $\zeta(s + \alpha)$ by a Dirichlet polynomial $\sum_{n \leq X} n^{-s-\alpha}$ with X substantially larger than T , say $X = T^\theta$ with $1 < \theta < 2$; the error in such an approximation is $\ll X^{1/2}/T$ when $t \approx T$ so that $\zeta(s)$ is well-approximated point-wise by this long Dirichlet polynomial. We introduce a smooth weight function $\psi(u)$ which is real and compactly supported, say on $[1, 2]$, and then evaluate the mean square of ζ by evaluating (as [GG] do)

$$\int_0^\infty \psi(t/T) |\zeta(1/2 + it)|^2 dt = \int_0^\infty \psi(t/T) \left| \sum_{n \leq X} \frac{1}{n^{1/2+it}} \right|^2 dt + O(X^{1/2} \log T).$$

Using coefficient correlations, Goldston and Gonek prove that this is

$$= \hat{\psi}(0) T \sum_{n \leq X} \frac{1}{n} + 2\Re T \int_y^\infty \left[\frac{v}{y} \right] \hat{\psi}(v) \frac{dv}{v} + O(T^\epsilon (X^{1/2} + X/T))$$

where

$$y = \frac{T}{2\pi X},$$

$[v]$ is the floor of v , i.e. the greatest integer less than or equal to v , and the Fourier transform is defined by

$$\hat{\psi}(v) = \int_{\mathbb{R}} \psi(u) e(uv) du$$

where $e(x) = \exp(2\pi i x)$.

On the other hand, the mean square of ζ is [A]

$$T \int_0^\infty \psi(t) \left(\log \frac{tT}{2\pi} + 2\gamma \right) dt + O(X^{1/2}).$$

Using the fact that

$$\sum_{n \leq X} \frac{1}{n} = \log X + \gamma + O(1/X)$$

it follows that

$$(1) \quad 2\Re \int_y^\infty \left[\frac{v}{y} \right] \hat{\psi}(v) \frac{dv}{v} \sim \int_0^\infty (\log uy + \gamma) \psi(u) du$$

as $y \rightarrow 0^+$. In the present formulation, this is purely a statement about a relationship between a function and its Fourier transform for real, compactly supported functions ψ .

We note that establishing the compatibility between the result of [GG] and that in [I, A] in the approach taken here relies on proving equation (1). This we do below, as a consequence of a more general identity.

2.1. General α and β . If we do this with small shifts α and β we have, again with $s = 1/2 + it$,

$$\begin{aligned} \int_{\mathbb{R}} \psi\left(\frac{t}{T}\right) \zeta(s + \alpha) \zeta(1 - s + \beta) dt &= T \int_{\mathbb{R}} \psi(t) \left(\zeta(1 + \alpha + \beta) + \left(\frac{tT}{2\pi}\right)^{-\alpha-\beta} \zeta(1 - \alpha - \beta) \right) dt \\ &\quad + O(T^{1/2+\epsilon}) \end{aligned}$$

by Ingham. By [GG] this is also equal to

$$\hat{\psi}(0)T \sum_{n \leq X} \frac{1}{n^{1+\alpha+\beta}} + 2T \int_y^\infty \sum_{h \leq \frac{v}{y}} \left(\frac{hT}{2\pi v}\right)^{-\alpha-\beta} \Re \hat{\psi}(v) \frac{dv}{v} + O(T^\epsilon(X/T + X^{1/2}))$$

where $y = \frac{T}{2\pi X}$. Since

$$\sum_{n \leq U} \frac{1}{n^{1+\alpha+\beta}} = \zeta(1 + \alpha + \beta) - \frac{U^{-\alpha-\beta}}{\alpha + \beta} + O(1/U)$$

for $\Re(\alpha + \beta) > 0$ and $U > T/\pi$, it follows that

$$\int_{\mathbb{R}} \psi(t) \left(\frac{y^{\alpha+\beta}}{\alpha + \beta} + t^{-\alpha-\beta} \zeta(1 - \alpha - \beta) \right) dt \sim 2\Re \int_y^\infty \sum_{h \leq \frac{v}{y}} h^{-\alpha-\beta} \hat{\psi}(v) \frac{dv}{v^{1-\alpha-\beta}}$$

as $y \rightarrow 0^+$. In fact, this is an identity. Note that (1) is the limiting case when $\alpha, \beta \rightarrow 0$ of this identity.

When we pass to higher moments the corresponding identities will be consequences of conjectures rather than facts. Thus, it is desirable to give a direct proof of this now, one that can be imitated later.

Proof. Write $\delta = \alpha + \beta$. We can extend the integration in v down to $v = 0$ since the integrand is 0 for $0 < v < y$. We will show that

$$(2) \quad \int_{\mathbb{R}} \psi(t) \left(\frac{y^\delta}{\delta} + t^{-\delta} \zeta(1 - \delta) \right) dt = 2\Re \int_0^\infty \sum_{h \leq \frac{v}{y}} h^{-\delta} \hat{\psi}(v) \frac{dv}{v^{1-\delta}}.$$

To begin with use

$$\sum_{h \leq \frac{v}{y}} h^{-\delta} = \frac{1}{2\pi i} \int_{(2)} \zeta(s + \delta) \left(\frac{v}{y} \right)^s \frac{ds}{s}.$$

Then write

$$2\Re \hat{\psi}(v) = \int_{\mathbb{R}} \psi(u) (e(-uv) + e(uv)) du.$$

Because of the support of ψ the u -integral only runs over a finite segment of the positive reals. We now want to interchange the u -integration and the v -integration, but we have to be careful here. Basically we first split the u -integrand into two pieces, one with $e(uv)$ and one with $e(-uv)$ and we deform the paths of the two ensuing v -integrals by rotating the straight line paths along the positive reals onto the positive imaginary axis (in the case of the $e(uv)$ integral) and the negative imaginary axis (in the case of the $e(-uv)$ integral). In this way we get that the right hand side of (2) is

$$\int_0^\infty \psi(u) \frac{1}{2\pi i} \int_{(2)} \zeta(s + \delta) \frac{y^{-s}}{s} \left(\int_0^\infty (iv)^{s+\delta} e^{-2\pi v u} \frac{dv}{v} + \int_0^\infty (-iv)^{s+\delta} e^{-2\pi v u} \frac{dv}{v} \right) ds du.$$

The sum of the two v -integrals is

$$2 \cos \frac{\pi(s + \delta)}{2} (2\pi)^{-s-\delta} \Gamma(s + \delta) u^{-s-\delta}$$

which is $u^{-s-\delta} \chi(1 - s - \delta)$ where χ is the factor from the functional equation $\zeta(1 - s) = \chi(1 - s) \zeta(s)$. Thus, our expression simplifies to

$$(3) \quad \int_0^\infty \psi(u) \frac{1}{2\pi i} \int_{(2)} \zeta(1 - s - \delta) \frac{y^{-s} u^{-s-\delta}}{s} ds du.$$

Now we move the s -path of integration to the left to $\Re s = -\infty$. When we account for the residues from the poles at $s = 0$ and $s = -\delta$ we exactly obtain the left-hand side of (2). \square

Note that we have shown that if $\Re s > 0$ then

$$(4) \quad 2 \int_0^\infty v^s \Re \hat{\psi}(v) \frac{dv}{v} = \chi(1 - s) \int_0^\infty \psi(t) t^{-s} dt$$

for functions ψ compactly supported on $(0, \infty)$.

We conclude that one can calculate the second moment of the zeta function with a power savings on the error term using long Dirichlet polynomials (i.e. longer than in the approximate functional equation), but this relies on establishing certain identities, such as that proved above. We shall see that this becomes more challenging for the higher moments.

3. FOURTH MOMENT

The fourth moment of the Riemann zeta-function has been much studied with the explicit formula of Motohashi [M] one of the crowning achievements.

The main goal of this paper is to revisit this topic but through the lens described in the last section. We want to understand how one can approach the fourth moment via the mean square of a long Dirichlet polynomial approximation to $\zeta(s)^2$. This point of view has also been explored to a certain extent in the work [H-B] of Heath-Brown, which built upon that of Ingham and Atkinson [I, A]; see also [CG]. But each of the above used an approximate functional equation to reduce the necessary length of the approximating polynomial. Here we want to see what happens purely with Dirichlet polynomials. This perspective will turn out to be useful in a future discussion of yet higher moments and divisor correlations.

4. THE 4TH MOMENT OF ZETA

To begin with we recall the fourth moment of zeta. Let $s = 1/2 + it$ and suppose that $\alpha, \beta, \gamma, \delta \ll (\log T)^{-1}$. Then

$$(5) \quad \int_0^\infty \psi\left(\frac{t}{T}\right) \zeta(s+\alpha)\zeta(s+\beta)\zeta(1-s+\gamma)\zeta(1-s+\delta) dt \\ = T \int_0^\infty \psi(t) \mathcal{Z}_{tT}(\alpha, \beta, \gamma, \delta) dt + O(T^{2/3+\epsilon}),$$

where

$$(6) \quad \mathcal{Z}_t(\alpha, \beta, \gamma, \delta) = Z(\alpha, \beta, \gamma, \delta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} Z(-\gamma, \beta, -\alpha, \delta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\delta} Z(-\delta, \beta, \gamma, -\alpha) \\ + \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} Z(\alpha, -\gamma, -\beta, \delta) + \left(\frac{t}{2\pi}\right)^{-\beta-\delta} Z(\alpha, -\delta, \gamma, -\beta) \\ + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} Z(-\gamma, -\delta, -\alpha, -\beta)$$

where

$$Z(\alpha, \beta, \gamma, \delta) = \frac{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)}$$

(Perhaps “recall” is not the correct word here since the above is certainly a theorem but its proof is not written down in full details anywhere.)

Now

$$\zeta(s + \alpha)\zeta(s + \beta) = \sum_{n=1}^{\infty} \frac{\tau_{\alpha,\beta}(n)}{n^{2s}}$$

where

$$\tau_{\alpha,\beta}(n) := \sum_{hk=n} h^{-\alpha} k^{-\beta}.$$

We let

$$D_{\alpha,\beta}(s) = \sum_{n=1}^{\infty} \frac{\tau_{\alpha,\beta}(n)}{n^s}$$

and

$$D_{\alpha,\beta}(s; X) = \sum_{n \leq X} \frac{\tau_{\alpha,\beta}(n)}{n^s}$$

be its approximating Dirichlet polynomial of length X . Let

$$I(T; X) = I_{\alpha,\beta,\gamma,\delta}(T; X) = \int_0^\infty \psi\left(\frac{t}{T}\right) D_{\alpha,\beta}(s, X) D_{\gamma,\delta}(1-s, X) dt.$$

Our basic question is: How does $I(T; X)$ approach $\int_0^\infty \psi\left(\frac{t}{T}\right) \zeta(s + \alpha)\zeta(s + \beta)\zeta(1-s + \gamma)\zeta(1-s + \delta) dt$ as $X \rightarrow \infty$?

5. DESCENDING THROUGH THE RECIPE

If we cheat, we can work backwards in a way and deduce the behavior of $I(T; X)$ from that of $\int_0^\infty \psi\left(\frac{t}{T}\right) \zeta(s + \alpha)\zeta(s + \beta)\zeta(1-s + \gamma)\zeta(1-s + \delta) dt$. The starting point is that by Perron's formula

$$D_{\alpha,\beta}(s; X) = \frac{1}{2\pi i} \int_{(2)} D_{\alpha,\beta}(s + z) X^z \frac{dz}{z}.$$

So, then

$$I(T; X) = \frac{1}{(2\pi i)^2} \iint_{z,w} X^{w+z} \int_0^\infty \psi\left(\frac{t}{T}\right) D_{\alpha,\beta}(s + z) D_{\gamma,\delta}(1-s + w) dt \frac{dz}{z} \frac{dw}{w}.$$

Now, if we move the paths of integration in z and w so that their real parts are positive but small, then we can use the formula (5) to evaluate the integral over t ; however the imaginary parts of z and w can be large. We believe (5) holds uniformly when the real parts of the shifts are $\ll 1/\log T$ but the imaginary parts of the shifts can be large, as large as $T^{1-\epsilon}$ and even larger. This has not been proven to hold for (5), but has been proven, as mentioned above, by Sandro Bettin [B] in the case of the second moment.

Now we use the fact that

$$D_{\alpha,\beta}(s + z) = D_{\alpha+z,\beta+z}(s)$$

and, assuming the uniform version of (5), replace $\int_0^\infty \psi\left(\frac{t}{T}\right) D_{\alpha,\beta}(s + z) D_{\gamma,\delta}(1-s + w) dt$ in the above by

$$T \int_0^\infty \psi(t) \mathcal{Z}_{tT}(\alpha + z, \beta + z, \gamma + w, \delta + w) dt.$$

Next, note that

$$\mathcal{Z}_t(\alpha + z, \beta + z, \gamma + w, \delta + w) = \mathcal{Z}_t(\alpha + z + w, \beta + z + w, \gamma, \delta)$$

follows just from the symmetries of \mathcal{Z}_t . We now change variables with $s = z + w$ and have

$$I(T; X) = \frac{1}{(2\pi i)^2} \int_{\Re s=4} \int_{\Re w=2} \frac{X^s}{w(s-w)} T \int_0^\infty \psi(t) \mathcal{Z}_{tT}(\alpha + s, \beta + s; \gamma, \delta) dt dw ds + O(T^{2/3+\epsilon}).$$

We move the path of integration in the w -variable off to the left, to $-\infty$ and collect the residue from the pole at $w = 0$. In this way we have

$$I(T; X) = \frac{1}{2\pi i} \int_{\Re s=4} \frac{X^s}{s} T \int_0^\infty \psi(t) \mathcal{Z}_{tT}(\alpha + s, \beta + s; \gamma, \delta) dt ds + O(T^{2/3+\epsilon}).$$

6. HOW MANY SWAPS TO USE?

In the formula (6) we sort the terms according to the number U of “swaps”; this is the number of pairs of variables that are exchanged in the arguments of \mathcal{Z}_t . Specifically, there are 6 terms on the right side of (6), the first has 0-swaps, the next four each have one swap, and the sixth and last term has two swaps. This notion is relevant because in the expressions for

$$\mathcal{Z}_t(\alpha + s, \beta + s; \gamma, \delta)$$

the terms with U swaps have factors of the form $\left(\frac{tT}{2\pi}\right)^{-Us}$. When paired with the factor X^s we see that to evaluate the relevant integral over s we should move the path to the right (and get 0 for an answer) whenever $X < \left(\frac{tT}{2\pi}\right)^U$; otherwise we should move the path to the left and evaluate the integral as the sum of the residues over the poles.

7. EXPLICIT FORMULA: 0-SWAPS

We refer to the terms with 0-swaps as the “diagonal” terms. These are just

$$\hat{\psi}(0) T \frac{1}{2\pi i} \int_{\Re s=4} \frac{X^s}{s} \mathcal{Z}(\alpha + s, \beta + s; \gamma, \delta) ds$$

which lead to $T\hat{\psi}(0)$ times

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{(2)} \frac{\zeta(s+1+\alpha+\gamma)\zeta(s+1+\alpha+\delta)\zeta(s+1+\beta+\gamma)\zeta(s+1+\beta+\delta)}{\zeta(2s+2+\alpha+\beta+\gamma+\delta)} \frac{X^s}{s} ds \\
&= \frac{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)} \\
&\quad - \frac{X^{-\alpha-\gamma}}{\alpha+\gamma} \frac{\zeta(1+\delta-\gamma)\zeta(1+\beta-\alpha)\zeta(1+\beta-\alpha+\delta-\gamma)}{\zeta(2+\beta-\alpha+\delta-\gamma)} \\
&\quad - \frac{X^{-\alpha-\delta}}{\alpha+\delta} \frac{\zeta(1+\gamma-\delta)\zeta(1+\beta-\alpha)\zeta(1+\beta-\alpha+\gamma-\delta)}{\zeta(2+\beta-\alpha+\gamma-\delta)} \\
&\quad - \frac{X^{-\beta-\gamma}}{\beta+\gamma} \frac{\zeta(1+\beta-\alpha)\zeta(1+\delta-\gamma)\zeta(1+\alpha-\beta+\delta-\gamma)}{\zeta(2+\alpha-\beta+\delta-\gamma)} \\
&\quad - \frac{X^{-\beta-\delta}}{\beta+\delta} \frac{\zeta(1+\alpha-\beta)\zeta(1+\gamma-\delta)\zeta(1+\alpha-\beta+\gamma-\delta)}{\zeta(2+\alpha-\beta+\gamma-\delta)}
\end{aligned}$$

plus an error term which is $O(T^{2/3+\epsilon})$.

8. EXPLICIT FORMULA: ONE SWAP

There are four terms with one-swap. One of these is

$$(7) \quad T \int_0^\infty \psi(t) \frac{1}{2\pi i} \int_{\Re s=4} \frac{X^s}{s} \left(\frac{tT}{2\pi} \right)^{-\alpha-\gamma-s} Z(-\gamma, \beta+s, -\alpha-s, \delta) ds dt.$$

If $X < \frac{T}{2\pi}$ this is 0. If $X > \frac{T}{2\pi}$ then we evaluate it as a sum over the residues at all of the poles (near $s=0$) of the integrand. Note that

$$Z(-\gamma, \beta+s, -\alpha-s, \delta) = \frac{\zeta(1-\alpha-\gamma-s)\zeta(1-\gamma+\delta)\zeta(1-\alpha+\beta)\zeta(1+\beta+\delta+s)}{\zeta(2-\alpha+\beta-\gamma+\delta)}$$

which has poles at $s = -\alpha - \gamma$ and $s = -\beta - \delta$. The sum of the residues at these poles and at $s=0$ give

$$\begin{aligned}
(8) \quad T \int_0^\infty \psi(t) & \left(\left(\frac{tT}{2\pi} \right)^{-\alpha-\gamma} \frac{\zeta(1-\alpha-\gamma)\zeta(1-\gamma+\delta)\zeta(1-\alpha+\beta)\zeta(1+\beta+\delta)}{\zeta(2-\alpha+\beta-\gamma+\delta)} \right. \\
& + \frac{X^{-\alpha-\gamma}}{\alpha+\gamma} \frac{\zeta(1-\gamma+\delta)\zeta(1-\alpha+\beta)\zeta(1-\alpha+\beta-\gamma+\delta)}{\zeta(2-\alpha+\beta-\gamma+\delta)} \\
& \left. - \frac{X^{-\beta-\delta}}{\beta+\delta} \left(\frac{tT}{2\pi} \right)^{-\alpha+\beta-\gamma+\delta} \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\zeta(1-\gamma+\delta)\zeta(1-\alpha+\beta)}{\zeta(2-\alpha+\beta-\gamma+\delta)} \right) dt.
\end{aligned}$$

Note that the term in the integrand which is independent of t cancels with a term from the 0-swap diagonal piece above.

9. EXPLICIT FORMULA: TWO SWAPS

For the term with two-swaps we have

$$T \int_0^\infty \psi(t) \frac{1}{2\pi i} \int_{(2)} \left(\frac{tT}{2\pi} \right)^{-\alpha-\beta-\gamma-\delta-2s} Z(-\gamma, -\delta, -\alpha-s, -\beta-s) \frac{X^s}{s} ds dt.$$

If $T^2 > \pi^2 X$ then the answer will be 0 as can be seen by moving the path of integration to the right. If $T^2 < 4\pi^2 X$ then we use the fact that

$$Z(-\gamma, -\delta, -\alpha-s, -\beta-s) = \frac{\zeta(1-\alpha-\gamma-s)\zeta(1-\beta-\gamma-s)\zeta(1-\alpha-\delta-s)\zeta(1-\beta-\delta-s)}{\zeta(2-\alpha-\beta-\gamma-\delta-2s)}$$

has poles at $s = \alpha + \gamma, \alpha + \delta, \beta + \gamma$ and $\beta + \delta$; so together with the residue from the pole at $s = 0$ we will get a total of five terms for this integral:

$$\begin{aligned} & \left(\frac{tT}{2\pi} \right)^{-\alpha-\beta-\gamma-\delta} \frac{\zeta(1-\alpha-\gamma)\zeta(1-\beta-\gamma)\zeta(1-\alpha-\delta)\zeta(1-\beta-\delta)}{\zeta(2-\alpha-\beta-\gamma-\delta)} \\ & + \frac{X^{-\alpha-\gamma}}{\alpha+\gamma} \left(\frac{tT}{2\pi} \right)^{\alpha+\beta+\gamma-\delta} \frac{\zeta(1+\alpha-\beta)\zeta(1+\gamma-\delta)\zeta(1+\alpha-\beta+\gamma-\delta)}{\zeta(2+\alpha-\beta+\gamma-\delta)} \\ & + \frac{X^{-\alpha-\delta}}{\alpha+\delta} \left(\frac{tT}{2\pi} \right)^{\alpha-\beta-\gamma+\delta} \frac{\zeta(1-\gamma+\delta)\zeta(1+\alpha-\beta-\gamma+\delta)\zeta(1+\alpha-\beta)}{\zeta(2+\alpha-\beta-\gamma+\delta)} \\ & + \frac{X^{-\beta-\gamma}}{\alpha+\delta} \left(\frac{tT}{2\pi} \right)^{-\alpha+\beta+\gamma-\delta} \frac{\zeta(1-\alpha+\beta)\zeta(1-\alpha+\beta+\gamma-\delta)\zeta(1+\gamma-\delta)}{\zeta(2-\alpha+\beta+\gamma-\delta)} \\ & + \frac{X^{-\beta-\delta}}{\beta+\delta} \left(\frac{tT}{2\pi} \right)^{-\alpha+\beta-\gamma+\delta} \frac{\zeta(1-\alpha+\beta-\gamma+\delta)\zeta(1-\gamma+\delta)\zeta(1-\alpha+\beta)}{\zeta(2-\alpha+\beta-\gamma+\delta)} \end{aligned}$$

Notice that the last term here is the negative of the last term in the explicit formula for one-swap. In the case that $X > \left(\frac{T}{2\pi}\right)^2$ we add the 0-swap, one swap, and two swap terms all together and all of the dependency on X disappears; we recover exactly the formula (5). This is to be expected since when $X > \left(\frac{T}{2\pi}\right)^2$ it is known that the Dirichlet polynomial $D_{\alpha,\beta}(s; X)$ is a good point-wise approximation to $\zeta(s+\alpha)\zeta(s+\beta)$, so the mean squares are the same.

10. ASCENDING THROUGH CONVOLUTION SUMS: THE DIAGONAL

Next we turn to what can be said about the mean-square $I(T; X)$ but now from the point of view of coefficient correlations. We call this the “ascending” perspective. We consider

$$I^\psi(T, X) = \int_0^\infty D_{\alpha,\beta}(s, X) D_{\gamma,\delta}(1-s, X) \psi(t/T) dt = T \sum_{m,n \leq X} \frac{\tau_{\alpha,\beta}(m) \tau_{\gamma,\delta}(n)}{\sqrt{mn}} \hat{\psi}(T \log(m/n)).$$

The simplest terms are the diagonal terms:

$$\hat{\psi}(0)T \sum_{n \leq X} \frac{\tau_{\alpha,\beta}(n)\tau_{\gamma,\delta}(n)}{n}.$$

The sum here may be evaluated by Perron's formula as

$$\sum_{n \leq X} \frac{\tau_{\alpha,\beta}(n)\tau_{\gamma,\delta}(n)}{n} = \frac{1}{2\pi i} \int_{(2)} \sum_{n=1}^{\infty} \frac{\tau_{\alpha,\beta}(n)\tau_{\gamma,\delta}(n)}{n^{1+s}} \frac{X^s}{s} ds.$$

But

$$\sum_{n=1}^{\infty} \frac{\tau_{\alpha,\beta}(n)\tau_{\gamma,\delta}(n)}{n^{1+s}} = \mathcal{Z}(\alpha + s, \beta + s; \gamma, \delta)$$

and so the diagonal terms are identical to the 0-swap terms already considered above.

11. ASCENDING: TYPE I OFF-DIAGONALS WITH SHIFTS

Now we are left with the terms

$$= T \sum_{\substack{m, n \leq X \\ m \neq n}} \frac{\tau_{\alpha,\beta}(m)\tau_{\gamma,\delta}(n)}{\sqrt{mn}} \hat{\psi}(T \log(m/n)).$$

We let $m = n + h$ and obtain (just as in [GG])

$$2T\Re \int_y^\infty \sum_{h \leq \frac{v}{y}} M' \left(\frac{hT}{2\pi v}, h \right) \hat{\psi}(v) \frac{dv}{v}$$

where the coefficient correlation is

$$\sum_{n \leq u} \tau_{\alpha,\beta}(n)\tau_{\gamma,\delta}(n+h) = M(u, h) + E(u, h).$$

We use the δ -method of [DFI] to get an expression for $M'(u, h)$, the average value of $\sum_{n \leq u} \tau_{\alpha,\beta}(n)\tau_{\gamma,\delta}(n+h)$. Namely,

$$\sum_{q=1}^{\infty} c_q(h) P_{\alpha,\beta}(u, q) P_{\gamma,\delta}(u+h, q)$$

where $c_q(h) = \sum_{\substack{d|h \\ d|q}} d\mu(q/d)$ is the Ramanujan sum and where $P_{\alpha,\beta}(u, d)$ is the average value of $\sum_{n \leq u} \tau_{\alpha,\beta}(n)e(n/d)$. The latter is just the sum of the residues near 1 of $\mathcal{D}_{\alpha,\beta}(s, 1/d)u^{s-1}$ where

$$\mathcal{D}_{\alpha,\beta}(s, 1/d) = \sum_{n=1}^{\infty} \tau_{\alpha,\beta}(n)e(n/d)n^{-s}.$$

In [C] Lemma 4, for example, it is shown that $\mathcal{D}_{\alpha,\beta}(s, 1/q) - q^{1-2s-\alpha-\beta} \mathcal{D}_{\alpha,\beta}(s)$ is entire. Thus,

$$P_{\alpha,\beta}(u, q) = \frac{1}{q} \zeta(1 - \alpha + \beta) u^{-\alpha} q^{\alpha-\beta} + \frac{1}{q} \zeta(1 + \alpha - \beta) u^{-\beta} q^{\beta-\alpha}$$

and

$$M'(u, h) = \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} \left(\zeta(1 - \alpha + \beta) u^{-\alpha} q^{\alpha - \beta} + \zeta(1 + \alpha - \beta) u^{-\beta} q^{\beta - \alpha} \right) \\ \times \left(\zeta(1 - \gamma + \delta) u^{-\gamma} q^{\gamma - \delta} + \zeta(1 + \gamma - \delta) u^{-\delta} q^{\delta - \gamma} \right).$$

We multiply out this expression and have 4 terms each of which can be expressed in terms of

$$M_{\alpha, \beta; \gamma, \delta} := \frac{\zeta(1 + \alpha + \beta) \zeta(1 + \gamma + \delta)}{\zeta(2 + \alpha + \beta + \gamma)} \sum_{d|h} \frac{1}{d^{1 + \alpha + \beta + \gamma + \delta}}.$$

Specifically, we have

$$M'(u, h) = u^{-\alpha - \gamma} M_{-\gamma, \beta; -\alpha, \delta} + u^{-\alpha - \delta} M_{-\delta, \beta; \gamma, -\alpha} + u^{-\beta - \gamma} M_{\alpha, -\gamma; -\beta, \delta} + u^{-\beta - \delta} M_{\alpha, -\delta; \gamma, -\beta}.$$

Now, much as we did in the first section we compute

$$2\Re \int_0^{\infty} \sum_{h \leq v/y} M'(\frac{hT}{2\pi v}, h) \hat{\psi}(v) \frac{dv}{v}.$$

(Note that the integrand is 0 for $0 < v < y$.) We have that one term (the term corresponding to $u^{-\alpha - \gamma} M_{-\gamma, \beta; -\alpha, \delta}$) of $\sum_{h \leq v/y} M'(\frac{hT}{2\pi v}, h)$ is

$$\frac{\zeta(1 - \alpha + \beta) \zeta(1 - \gamma + \delta)}{\zeta(2 - \alpha + \beta - \gamma + \delta)} \left(\frac{T}{2\pi v} \right)^{-\alpha - \gamma} \sum_{hd \leq v/y} h^{-\alpha - \gamma} d^{-1 - \beta - \delta} \\ = \frac{\zeta(1 - \alpha + \beta) \zeta(1 - \gamma + \delta)}{\zeta(2 - \alpha + \beta - \gamma + \delta)} \left(\frac{T}{2\pi v} \right)^{-\alpha - \gamma} \frac{1}{2\pi i} \int_{(2)} \zeta(s + \alpha + \gamma) \zeta(s + 1 + \beta + \delta) \frac{(\frac{v}{y})^s}{s} ds.$$

As before, see (3), we compute that

$$\zeta(s + \alpha + \gamma) \int_0^{\infty} \hat{\psi}(v) v^{s - 1 + \alpha + \gamma} dv = \zeta(1 - s - \alpha - \gamma) \int_0^{\infty} \psi(u) u^{-s - \alpha - \gamma} du.$$

Then, by moving the s -path of integration to $\Re s = -1/2$ we have

$$\frac{1}{2\pi i} \int_{(2)} \zeta(1 - s - \alpha - \gamma) \zeta(s + 1 + \beta + \delta) \frac{y^{-s} u^{-s - \alpha - \gamma}}{s} ds \\ = u^{-\alpha - \gamma} \zeta(1 - \alpha - \gamma) \zeta(1 + \beta + \delta) + \zeta(1 - \alpha + \beta - \gamma + \delta) \frac{y^{\alpha + \gamma}}{\alpha + \gamma} \\ - \zeta(1 - \alpha + \beta - \gamma + \delta) \frac{y^{\beta + \delta} u^{-\alpha + \beta - \gamma + \delta}}{\beta + \delta} + O(y^{1/2}).$$

Now we have shown that

$$\begin{aligned}
2\Re \int_0^\infty \sum_{h \leq v/y} M'(\frac{hT}{2\psi v}, h) \hat{\psi}(v) \frac{dv}{v} &= \int_0^\infty \psi(u) \left(\frac{T}{2\pi}\right)^{-\alpha-\gamma} \frac{\zeta(1-\alpha+\beta)\zeta(1-\gamma+\delta)}{\zeta(2-\alpha+\beta-\gamma+\delta)} \\
&\times \left(u^{-\alpha-\gamma} \zeta(1-\alpha-\gamma)\zeta(1+\beta+\delta) + \zeta(1-\alpha+\beta-\gamma+\delta) \frac{y^{\alpha+\gamma}}{\alpha+\gamma} \right. \\
&\quad \left. - \zeta(1-\alpha+\beta-\gamma+\delta) \frac{y^{\beta+\delta} u^{-\alpha+\beta-\gamma+\delta}}{\beta+\delta} \right) du + O(y^{1/2}).
\end{aligned}$$

Recalling that $y = T/(2\pi X)$ we see that this term matches up with the first term (8) that we got from the recipe with one swap. Similarly the other three terms match up. It follows that the one-swap terms match up with the Type I off-diagonal terms.

12. CONCLUSION

The coefficient correlation terms that we have analyzed here contribute to $I(T; X)$ once X gets larger than T . We have shown two very different ways to view these terms: from the 1-swap terms of the recipe conjecture, and from the Type-I divisor correlation sums much studied in analytic number theory. Our results illustrate that there are other terms to be considered once $X > T^2$, i.e. that there is something missing from the above analysis. These are the divisor correlations that must match the 2-swapped terms.

The next step is to describe, from the ascending perspective, the terms that we see in the descent from (5) to the terms with 2-swaps. This will be the subject of a subsequent paper where the coefficient sums to be considered are much as in [BK]:

$$\sum_{\substack{M_1, M_2, h_1, h_2 \\ (M_1, M_2)=1}} \sum_{\substack{m_1, m_2, n_1, n_2 \\ (*_1), (*_2) \\ m_1 m_2 \leq X}} \frac{m_1^{-\alpha} n_1^{-\gamma} m_2^{-\beta} n_2^{-\delta}}{m_1 m_2} \hat{\psi} \left(T \left(\frac{h_1}{m_1 M_1} + \frac{h_2}{m_2 M_2} \right) \right)$$

where

$$(*_1) : M_1 m_1 = M_2 n_1 + h_1 \quad \quad (*_2) : M_2 m_2 = M_1 n_2 + h_2$$

for coprime integers M_1 and M_2 .

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